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# On two-dimensional surface attractors and repellers on 3-manifolds\*

V. Grines      V. Medvedev      E. Zhuzhoma<sup>†</sup>

## Abstract

We show that if  $f : M^3 \rightarrow M^3$  is an  $A$ -diffeomorphism with a surface two-dimensional attractor or repeller  $\mathcal{B}$  and  $M_{\mathcal{B}}^2$  is a supporting surface for  $\mathcal{B}$ , then  $\mathcal{B} = M_{\mathcal{B}}^2$  and there is  $k \geq 1$  such that:

- 1)  $M_{\mathcal{B}}^2$  is a union  $M_1^2 \cup \dots \cup M_k^2$  of disjoint tame surfaces such that every  $M_i^2$  is homeomorphic to the 2-torus  $T^2$ .
- 2) the restriction of  $f^k$  to  $M_i^2$  ( $i \in \{1, \dots, k\}$ ) is conjugate to Anosov automorphism of  $T^2$ .

## 1 Introduction

One of the important question of Dynamical Systems Theory is the relationship between a fixed class of systems under consideration and the topology of underlying manifolds. This question is closely connected with a structure of non-wandering set of a dynamic system. For example, Franks [9] and Newhouse [19] have shown that any codimension one Anosov diffeomorphism is conjugate to a hyperbolic torus automorphism (as a consequence, a manifold admitting such diffeomorphisms is homeomorphic to the torus  $T^n$ ). A simple proof of this Franks-Newhouse theorem that uses foliation theory techniques was obtained in [12].

Recently Grines and Zhuzhoma [10] proved that if a closed  $n$ -manifold  $M^n$ ,  $n \geq 3$ , admits a structurally stable diffeomorphism  $f$  with an orientable codimension one expanding attractor, then  $M^n$  is homotopy equivalent to

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the  $n$ -torus  $T^n$  and is homeomorphic to  $T^n$  for  $n \neq 4$ . Moreover, there are no nontrivial basic sets different from the codimension one expanding attractor in nonwandering set of the diffeomorphism  $f$ . This allowed to them to classify, up to conjugacy, structurally stable diffeomorphisms having codimension one expanding attractors and contracting repellers on  $T^n$ .

A key point in the mentioned above results is the existence of so-called hyperbolic structure on a non-wandering set. More precisely, let  $f : M \rightarrow M$  be a diffeomorphism of a closed  $m$ -manifold  $M$ ,  $m = \dim M \geq 2$ , endowed with some Riemannian metric  $\rho$  (all definitions in this section can be found in [16] and [24], unless otherwise indicated). Recall that a point  $x \in M$  is *non-wandering* if for any neighborhood  $U$  of  $x$ ,  $f^n(U) \cap U \neq \emptyset$  for infinitely many integers  $n$ . Then the non-wandering set  $NW(f)$ , defined as the set of all non-wandering points, is an  $f$ -invariant and closed set. A closed invariant set  $\Lambda \subset M$  is *hyperbolic* if there is a continuous  $f$ -invariant splitting of the tangent bundle  $T_\Lambda M$  into stable and unstable bundles  $E_\Lambda^s \oplus E_\Lambda^u$  with

$$\|df^n(v)\| \leq C\lambda^n\|v\|, \quad \|df^{-n}(w)\| \leq C\lambda^n\|w\|, \quad \forall v \in E_\Lambda^s, \forall w \in E_\Lambda^u, \forall n \in \mathbb{N},$$

for some fixed  $C > 0$  and  $\lambda < 1$ . For each  $x \in \Lambda$ , the sets  $W^s(x) = \{y \in M : \lim_{j \rightarrow \infty} \rho(f^j(x), f^j(y)) \rightarrow 0\}$ ,  $W^u(x) = \{y \in M : \lim_{j \rightarrow \infty} \rho(f^{-j}(x), f^{-j}(y)) \rightarrow 0\}$  are smooth, injective immersions of  $E_x^s$  and  $E_x^u$  that are tangent to  $W_x^s, W_x^u$  respectively.  $W^s(x), W^u(x)$  are called *stable* and *unstable manifolds* at  $x$ .

An important class of dynamical systems is made up of the diffeomorphisms satisfying Smale's Axiom A [26], the so-called *A-diffeomorphisms*. Given such a diffeomorphism  $f$ , its recurrent behavior is captured in its non-wandering set  $NW(f)$ , which can be decomposed into invariant topologically transitive pieces. To be precise, a diffeomorphism  $f : M \rightarrow M$  is an *A-diffeomorphism* if its non-wandering set  $NW(f)$  is hyperbolic and the periodic points are dense in  $NW(f)$ . According to Smale's Spectral Decomposition Theorem,  $NW(f)$  is decomposed into finitely many disjoint so-called basic sets  $\Omega_1, \dots, \Omega_k$  such that each  $\Omega_i$  is closed,  $f$ -invariant and contains a dense orbit [26]. Following [1], the pair  $(a, b)$  is said to be a *type* of basic set  $\Omega$  if  $a = \dim E_x^s$  and  $b = \dim E_x^u$ , where  $x \in \Omega$ .

S. Smale posed several kinds of basic sets:

- (a) zero dimensional ones such as Smale's horseshoe;
- (b) one-dimensional ones such as so-called Smale solenoids and its generalization Smale-Williams solenoids (see [27], the name is suggested in [20]);

(c) codimension one expanding attractors or contracting repellers of DA-diffeomorphisms;

(d) basic sets of transitive Anosov diffeomorphisms whose dimension equals to the dimension of underlying manifold because they coincide with the manifold.

It is well known that there is no restriction on the topology of underlying manifold when a basic set  $\mathcal{B}$  is zero dimensional [25].

But in the case of non zero dimensional the situation is other in general (see above mentioned examples). In addition, we would like to notice the article [5] in which was proved that a closed orientable 3-manifold  $M$  contains a knotted Smale solenoid if and only if  $M$  has a lens space  $L(p, q)$ , with  $p \neq 0, \pm 1$ , as a prime factor <sup>1</sup>. This result was repeated recently in [14].

This paper is devoted to topological classification of so-called surface basic set of  $A$ -diffeomorphisms on smooth closed orientable 3-manifolds. Let us introduce a concept of surface basic set.

**Definition 1** *A basic set  $\mathcal{B}$  of an  $A$ -diffeomorphism  $f : M^3 \rightarrow M^3$  is called surface basic set if  $\mathcal{B}$  belongs to an  $f$ -invariant closed surface  $M_{\mathcal{B}}^2$  topologically embedded in the 3-manifold  $M^3$ .*

The  $f$ -invariant surface  $M_{\mathcal{B}}^2$  is called a *supporting surface* for  $\mathcal{B}$ .

By definition, a supporting surface is not necessary connected. But it is obviously that there is some power of diffeomorphism  $f$  for which every surface basic set has connected supporting surface.

Let us recall that a basic set  $\mathcal{B}$  of  $A$ -diffeomorphism  $f : M \rightarrow M$  is called an *attractor* if there is a closed neighborhood  $U$  of  $\mathcal{B}$  such that  $f(U) \subset \text{int } U$ ,  $\bigcap_{j \geq 0} f^j(U) = \mathcal{B}$ .

If  $\mathcal{B}$  is a two-dimensional basic set of  $A$ -diffeomorphism  $f$  on a closed 3-manifold  $M^3$  then, accordingly to [22] (theorem 3),  $\mathcal{B}$  is either an attractor or repeller.

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<sup>1</sup>Let  $M$ ,  $M_1$ ,  $M_2$  are connected 3-manifolds. We recall that  $M$  is a connected sum of  $M_1$  and  $M_2$  and denote this  $M = M_1 \# M_2$  if there are 3-cells  $B_i \subset M_i$  and embeddings  $h_i : M_i - \text{Int} B_i \rightarrow M$  ( $i = 1, 2$ ) with  $h_1(M_1 - \text{Int} B_1) \cap h_2(M_2 - \text{Int} B_2) = h_1(\partial B_1) = h_2(\partial B_2)$  and  $M = h_1(M_1 - \text{Int} B_1) \cup h_2(M_2 - \text{Int} B_2)$ .  $M_1$ ,  $M_2$  are called factors. A connected 3-manifold  $M$  is called prime manifold if from condition  $M = M_1 \# M_2$  it follows that exactly one manifold from  $M_1$  and  $M_2$  must be prime. It is well known (see, for example, [11], theorem 3.15) that each compact 3-manifold can be expressed as a connected sum of finite number of prime factors. Let us recall that a lens space is prime (see [11], ex. 3.12).

Recall that an attractor is called an *expanding attractor* if topological dimension  $\dim \mathcal{B}$  of  $\mathcal{B}$  is equal to the dimension  $\dim(E_{\mathcal{B}}^u)$  of the unstable bundle  $E_{\mathcal{B}}^u$  (the name is suggested in [27], [28]). A contracting repeller of a diffeomorphism  $f$  is an expanding attractor for  $f^{-1}$ . Certainly, one can consider a contracting repeller instead of an expanding attractor, and vice versa. It is well known that a codimension one expanding attractor consists of the  $(\dim M - 1)$ -dimensional unstable manifolds  $W^u(x)$ ,  $x \in \mathcal{B}$ , and is locally homeomorphic to the product of  $(\dim M - 1)$ -dimensional Euclidean space and a Cantor set (see, for example, [22], theorem 2). The similar structure has codimension one contracting repeller. Therefore one use the notion *pseudotame basic set*, meaning an expanding attractor or contracting repeller.

As we know, the next Smale's question is open (see [26], p. 785): is there codimension one basic set that is not compact submanifolds and is not locally product of Euclidean space and a Cantor set.

In section 3 we shall prove (in the lemma 1) that surface two-dimensional attractor (repeller)  $\mathcal{B}$  of an  $A$ -diffeomorphism  $f : M^3 \rightarrow M^3$  has type  $(2, 1)$   $((1, 2))$ . It follows from there that  $\mathcal{B}$  is neither expanding attractor nor contracting repeller. Moreover, we will prove in section 3 (lemma 2) that a two-dimensional basic set  $\mathcal{B}$  coincide with its supporting surface  $M_{\mathcal{B}}^2$ .

Thus, Smale's question in the case under consideration can be formulated as follows: is there two-dimensional attractor (repeller) which has the type  $(2, 1)$   $((1, 2))$  and different from compact submanifold?

Let us notice that according to [15] there is an example of  $A$ -diffeomorphism of closed three-manifold such that its non-wandering set contains a two-dimensional surface basic set whose supporting surface is an essentially non-smoothly embedded two-torus.

We notice also that according to [21], [4] there is a Morse-Smale diffeomorphism  $f : S^3 \rightarrow S^3$  with the  $f$ -invariant attracting surface  $S$  homeomorphic to the sphere  $S^2$  that is wildly embedded into  $S^3$  (it is necessary to emphasize that  $S$  is not a basic set of  $f$  in this case).

However the first result of our paper claims that a supporting surface for surface basic set is a union of tame tori.

Let us recall the concept of a tame surface embedded in  $M^3$ .

Let  $D_0 : \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq 1, z = 0\}$  be standard disk and  $M^2$  a surface embedded in a three-manifold  $M^3$ .

A surface  $M^2$  is called *locally flat* or *tame* if for any point  $x \in M^2$  there is a neighborhood  $U_x$  of the point  $x$  in  $M^3$  and homeomorphism  $h_x : \overline{U_x} \rightarrow \mathbb{R}^3$

such that  $h(\overline{U_x \cap M^2}) = D_0$ .

**Theorem 1** *Let  $f : M^3 \rightarrow M^3$  be an  $A$ -diffeomorphism with the surface two-dimensional basic set  $\mathcal{B}$  and  $M_{\mathcal{B}}^2$  is a supporting surface for  $\mathcal{B}$ . Then  $\mathcal{B} = M_{\mathcal{B}}^2$  and there is a number  $k \geq 1$  such that  $M_{\mathcal{B}}^2$  is a union  $M_1^2 \cup \dots \cup M_k^2$  of disjoint tame surfaces such that every  $M_i^2$  is homeomorphic to the 2-torus  $T^2$ .*

The next theorem explains the dynamics of restriction of the diffeomorphism  $f$  to a surface basic set.

**Theorem 2** *Let the condition of theorem 1 are fulfilled, then there is number  $k \geq 1$  such that the restriction  $f^k$  to  $M_i^2$  ( $i \in \{1, \dots, k\}$ ) is conjugate to Anosov automorphism of  $T^2$ .*

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## 2 Proof of theorem 1

*Proof of theorem 1* follows from the next lemmas 1, 2 and 3 which will be proved below.

**Lemma 1** *Let  $\mathcal{B}$  be a two-dimensional surface attractor (repeller) of  $A$ -diffeomorphism  $f$  of three-manifold  $M^3$ . Then  $\mathcal{B}$  has type  $(2, 1)$  (resp.  $(1, 2)$ ).*

*Proof.* Suppose for definiteness that  $\mathcal{B}$  is a surface attractor and  $M_{\mathcal{B}}^2$  is the supporting surface for  $\mathcal{B}$ . According to [22] (theorem 1), the unstable manifold  $W^u(x)$  belongs to  $\mathcal{B}$  for any point  $z \in \mathcal{B}$ . Since the restriction of  $f$  to  $\mathcal{B}$  is transitive,  $\dim W^u(z)$  does not depend on the choice of  $z \in \mathcal{B}$ . Let us notice that  $\dim E_z^u = \dim W^u(z)$ . Thus it is sufficiently to prove that  $\dim W^u(z) = 1$ .

Suppose the contrary. It follows from  $\dim W^u(z) \leq \dim \mathcal{B}$  that either  $\dim W^u(z) = 0$  or  $\dim W^u(z) = 2$ .

If  $\dim W^u(z) = 0$  then  $\mathcal{B}$  would be an attracting orbit of the diffeomorphism  $f$  and, hence, a zero-dimensional basic set. It contradicts to supposition that  $\dim \mathcal{B} = 2$ .

Suppose that  $\dim W^u(z) = 2$ . As  $\mathcal{B}$  is nontrivial basic set, then it contains infinite set  $Per(f)_{\mathcal{B}}$  of periodic points which are dense in  $\mathcal{B}$ . From another hand, as  $\mathcal{B}$  is a surface basic set, then for any point  $p \in Per(f)_{\mathcal{B}}$  the unstable manifold  $W^u(p)$  belongs to the surface  $M_{\mathcal{B}}^2$  and consequently the set  $W^u(p) \setminus \{p\}$  does not contain periodic points of the diffeomorphism  $f$ . It contradicts to the fact that the set  $Per(f)_{\mathcal{B}}$  is dense in  $\mathcal{B}$ .  $\square$

Let  $\mathcal{B}$  be a two-dimensional attractor of  $A$ -diffeomorphism  $f : M^3 \rightarrow M^3$ .

According to D.V. Anosov [2] and R. Bowen [6] there is number  $k \geq 1$  such that the basic set  $\mathcal{B}$  can be represented as the union of disjoint closed set  $\mathcal{B}_1, \dots, \mathcal{B}_k$  such that  $f(\mathcal{B}_i) = \mathcal{B}_{i+1}$  ( $\mathcal{B}_{k+1} = \mathcal{B}_1$ ) and for any point  $z \in \mathcal{B}_i$   $W^u(z) = \mathcal{B}_i$ . It follows from the proof of lemma 1 that  $\dim W^u(z) = 1$  for any point  $z \in \mathcal{B}_i$  and  $W^u(z)$  belongs to  $\mathcal{B}_i$ .

Denote by  $F_i^u$  the family of unstable manifolds  $W^u(z)$  for all points  $z \in \mathcal{B}_i$ .

**Lemma 2** *Let  $f : M^3 \rightarrow M^3$  be an  $A$ -diffeomorphism whose non-wandering set contains the surface two-dimensional attractor  $\mathcal{B}$  with the supporting surface  $M_{\mathcal{B}}^2$ . Then:*

- 1)  $\mathcal{B} = M_{\mathcal{B}}^2$ ;
- 2) the family  $F_i^u$  is a continuous foliation without singularities on  $M_i^2$ .
- 3)  $M_{\mathcal{B}}^2$  is the union  $M_1^2 \cup \dots \cup M_k^2$  of disjoint surfaces such that each of them is homeomorphic to the 2-torus  $T^2$  and  $\mathcal{B}_i = M_i^2$ .

*Proof.* Put  $g = f^k$ . It is obviously that for any  $i \in \{1, \dots, k\}$  there is a surface  $M_i^2 \subseteq M^2$  which is supporting surface for the basic set  $\mathcal{B}_i$  of the diffeomorphism  $g$ . Let us show that  $\mathcal{B}_i = M_i^2$ .

As by assumption  $\dim \mathcal{B}_i = 2$  and  $\mathcal{B}_i \subset M_i^2$ , then  $\mathcal{B}_i$  contains a non-empty open subset, say  $V$ , in the interior topology of  $M_i^2$  (see thm. 4.3 [13]).

Let  $z$  be any point belonging to  $int V$ . According to [26], there is  $\alpha > 0$  such that the point  $z$  has a closed neighborhood  $U_z \subset V$  which is homeomorphic to the direct product  $\check{W}_{\alpha}^s(z) \times W_{\alpha}^u(z)$ , where  $\check{W}_{\alpha}^s(z) = \mathcal{B} \cap W_{\alpha}^s(z)$  and  $W_{\alpha}^s(z)$ ,  $W_{\alpha}^u(z)$  are a closed  $\alpha$ -neighborhoods of the point  $z$  in the some initial metric in the manifold  $W^s(z)$ ,  $W^u(z)$  respectively. It means that for any point  $w \in U_z$  there is a unique pair of points  $w^s \in \check{W}_{\alpha}^s(z)$ ,  $w^u \in W_{\alpha}^u(z)$  such that  $w = W^u(w^s) \cap W_{\alpha}^s(w^u)$ . Let us define the projection  $\pi_z : U_z \rightarrow W^s(z)$  as follows. For  $w \in U_z$ , put  $\pi_z(w) = w^s$ .

Since  $V$  is an open subset of  $M^2$  and  $M^2$  is topologically embedded in  $M^3$ , one can assume, without loss of generality, that  $U_z$  is homeomorphic to a closed 2-disk.

Put  $U_z^s = \pi_z(U_z)$ . Let us show that there is closed simple way  $\gamma \subset U_z^s$  such that  $z \in \text{int } \gamma$ .

It is well known (see, for example, [18]) that any simple curve is tame on a disk. Thus there is an open neighborhood  $D_z \subset U_z$  of the point  $z$  such that:

- 1) the set  $W^u(z) \cap D_z$  consists of exactly one connected component say  $\lambda \subset W^u(z)$ ;
- 2) the set  $D \setminus \lambda$  consists of two connected components say  $D_1, D_2$ .
- 3)  $\pi_z(D_1) \cap \pi_z(D_2) = \emptyset$ .

Let us choose any points  $z_1 \in \pi_z(D_1)$ ,  $z_2 \in \pi_z(D_2)$ . As the sets  $D_1 \cup \lambda$ ,  $D_2 \cup \lambda$  are path connected subsets and  $\pi_z$  is continuous map then  $\pi_z(D_1 \cup \lambda)$ ,  $\pi_z(D_2 \cup \lambda)$  are also path connected subsets. Moreover as  $\pi_z(D_1 \cup \lambda)$ ,  $\pi_z(D_2 \cup \lambda)$  are Hausdorff subspaces (in induced topology), then there is a simple arc  $\gamma_1 \subset \pi(D_1 \cup \lambda)$  joining the points  $z_1, z$  and there is a simple arc  $\gamma_2 \subset \pi_z(D_2 \cup \lambda)$  joining the points and  $z_2, z$ . Then the arc  $\gamma = \gamma_1 \cup \gamma_2$  is a simple arc joining the points  $z_1$  and  $z_2$ . By construction  $z \in \text{int } \gamma$ .

Introduce a parameter  $t \in [-1, 1]$  on the arc  $\gamma$  such that  $\gamma(-1) = z_1$ ,  $\gamma(0) = z$  and  $\gamma(1) = z_2$ . For any  $t \in [-1, 1]$  denote  $l_t^u = W_\alpha^u(\gamma(t))$  and put  $F_z^u = \bigcup_{t \in [-1, 1]} l_t^u$ .

For any point  $\gamma(t)$ , the unstable manifold  $W_\alpha^u(\gamma(t))$  intersects transversally the stable manifold  $W_\alpha^s(z)$  at a unique point.

Let  $V_0(z)$  be the disk consisting of the all points belonging to the union curves  $l_t^u$  ( $t \in [-1, 1]$ ). Due to the theorem on continuous dependence of unstable manifolds on initial conditions (see, for example, [26]), the family of curves  $F_z^u$  is locally homeomorphic to a family of straight lines parallel to the axis  $Ox$  on Euclidean plain by means some homeomorphism  $h_z : V_0(z) \rightarrow \mathbb{R}^2$ . Hence,  $F_z^u$  is a continuous foliation on the closed disk  $V_0(z)$ .

Let  $y \in \mathcal{B}_i$  be any point. As  $\overline{W^u(y)} = \mathcal{B}_i$ , then there is a point  $w \in \text{int } \gamma$  such that  $[y, w]^u \subset W^u(y)$ . Due to the theorem on continuous dependence of unstable manifolds on initial condition, there is an open neighborhood

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<sup>2</sup>See for example [7], proposition 6.3.12 (a) which claims that a Hausdorff space  $X$  is pathwise connected if and only if for every pair  $x_1, x_2$  ( $x_1 \neq x_2$ ) there exists a homeomorphic embedding  $h : I \rightarrow X$  of the closed unit interval in the space  $X$  satisfying  $h(0) = x_1, h(1) = x_2$  (e.i.,  $X$  is arcwise connected).



$U_w \subset \text{int } \gamma$  of the point  $w$  and an open neighborhood  $U_y \subset M_i^2$  of the point  $y$  such that for any point  $y' \in U_y$  there is the point  $w' \in U_w$  such that  $[w', y']^u \subset W^u(w')$  and consequently the point  $y'$  belongs to  $\mathcal{B}_i$ . It means that the set  $\mathcal{B}_i$  is open. As  $\mathcal{B}_i$  is closed then it coincides with the surface  $M_i^2$ . As  $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$  for  $i \neq j$  then also  $M_i^2 \cap M_j^2 = \emptyset$ .

Thus  $\mathcal{B} = M_{\mathcal{B}}^2$  and  $M_{\mathcal{B}}^2$  is the union  $M_1^2 \cup \dots \cup M_k^2$  of disjoint surfaces.

It follows from above arguments that for any point of  $b \in \mathcal{B}_i$  there is a neighborhood  $U_b$  and homeomorphism  $h_b : U_b \rightarrow \mathbb{R}^2$  such that  $h_b$  maps the intersection of curves from  $F_i^u$  with  $U_b$  on the family of straight lines which are parallel to the axis  $Ox$ . Thus family  $F_i^u$  is continuous transitive foliation without singularities on  $M_i^2$ .

Consequently the surface  $M_i^2$  is homeomorphic to either the Klein bottle or the torus. According to [17] any foliation without singularities on Klein bottle must have at least one closed leaf. As a consequence the Klein bottle does not admit transitive foliation. Thus the manifold  $M_i^2$  is homeomorphic to the torus and the lemma is completely proved.  $\square$

**Lemma 3** *The surface  $M_i^2$  is tame.*

*Proof.* According to the proof of lemma 2, for any point  $z \in M_i^2$  there exists a neighborhood  $V_0(z) \subset M_i^2$  such that:

- 1)  $V_0(z)$  is homeomorphic to the direct product  $\gamma \times W_\alpha^u(z)$ , where  $\gamma$  is simple curve belonging to  $W_\alpha^s(z)$ ;
- 2)  $V_0(z)$  is the union of the smooth curves  $W_\alpha^u(w)$ ,  $w \in \gamma$ , which belong to leaves of the foliation  $F_i^u$  and form the foliation  $\tilde{F}^u$  which is given on the neighborhood  $V_0(z)$ ;
- 3) any curve  $W_\alpha^u(w)$  intersects  $W_\alpha^s(z)$  in exactly one point;
- 4) the curve  $\gamma$  is a local section (in topological sense) for the foliation  $\tilde{F}^{u3}$ .

Let us show that there are a neighborhood  $B_z$  of the point  $z$  that is homeomorphic to 3-disk and an embedding  $h_z : B_z \rightarrow \mathbb{R}^3$  such that  $h_z(B_z \cap V_0(z)) = D_0$ .

Without lost of generality we can suppose that there exists a neighborhood  $\tilde{B}_z$  (of the point  $z$ ) which is homeomorphic to 3-disk such that:

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<sup>3</sup>We speak that  $\gamma$  is a local section (in topological sense) for the foliation  $\tilde{F}^u$  if for any point  $d \in \gamma$  there is neighborhood  $V_d \subset V_0(z)$  such that for any leaf  $l \subset \tilde{F}^u$  with  $(l \cap V_d) \neq \emptyset$  the intersection  $(l \cap V_d) \cap \gamma$  consists of exactly one point  $x_l$  and  $V_d \setminus \gamma$  consists of two component each of which contains exactly one component of the set  $l \setminus \{x_l\}$

- 1)  $W_\alpha^s(z) \subset \tilde{B}_z$ ,  $V_0(z) \subset \tilde{B}_z$ ;
- 2) there is diffeomorphism  $g : \overline{\tilde{B}_z} \rightarrow B_0^3$  such that  $g(W_\alpha^s(z)) \subset Oxy$ ,  $g(z) = O(0,0,0)$ , where  $B_0^3$  is a closed unit ball in  $\mathbb{R}^3$ .

Put  $V = g(V_0(z))$ ,  $Q = g(W_\alpha^s(z))$  and denote by  $\hat{F}^u$  the foliation (on  $V$ ) which is the image of the foliation  $F^u$  under the map  $g$ . By construction the curve  $\lambda = g(\gamma)$  is a local section (in topological sense) for the foliation  $\hat{F}^u$ . Let us choose closed simple arc  $\lambda_1 \subset \text{int } \lambda$ . Denote by  $a, b$  the endpoints of the arc  $\lambda_1$  and denote by  $a_0$  ( $b_0$ ) the point belonging to the connected component of the set  $\lambda \setminus \text{int } \lambda_1$  for which the point  $a$  ( $b$ ) belongs to its boundary. Denote by  $\gamma_{a_0a}$  and  $\gamma_{bb_0}$  the closed simple ARCS belonging to  $\lambda$  for which  $a_0, a$  and  $b, b_0$  are the endpoints respectively.

Let us show that there exist the simple piecewise linear arcs  $\tilde{\gamma}_{a_0a}$  and  $\tilde{\gamma}_{bb_0}$  belonging to  $Q$  with endpoints  $a_0, a$  and  $b, b_0$  respectively and satisfying the next conditions:

- 1)  $\tilde{\gamma}_{a_0a} \cap \tilde{\gamma}_{bb_0} = \emptyset$ ;
- 2)  $\tilde{\gamma}_{a_0a} \cap \lambda_1 = a$ ,  $\tilde{\gamma}_{bb_0} \cap \lambda_1 = b$
- 3) the endpoints of each linear link of the curves  $\tilde{\gamma}_{a_0a}$  ( $\tilde{\gamma}_{bb_0}$ ) belong to the curve  $\gamma_{a_0a}$  ( $\gamma_{bb_0}$ ).

Let us show the existence of the curve  $\gamma_{a_0a}$  (the existence of the curve  $\gamma_{bb_0}$  may be shown similarly). Put  $\nu = \gamma_{a_0a} \setminus \{a\}$  and choose an open neighborhood  $U_\nu \subset V$  of the set  $\nu$  such that

- 1)  $U_\nu \cap (\lambda_1 \cup \gamma_{bb_0}) = \emptyset$ ;
- 2)  $U_\nu$  admits a triangulation  $\Sigma = \bigcup_{i \in \mathbb{Z}^+} \sigma_i$  such that for any point  $x \in U_\nu$

any neighborhood  $U_x \subset U_\nu$  of the point  $x$  intersects only finite number of simplexes from the union  $\bigcup_{i \in \mathbb{Z}^+} \sigma_i$ .

Introduce a parameter  $t \in [0, \infty)$  on the arc  $\nu$  such that  $\nu(0) = a_0$  and  $\nu(t)$  tends to  $a$  as  $t \rightarrow +\infty$ .

Since  $\nu(t)$  tends to  $a$  as  $t \rightarrow +\infty$ , there is the sequence of numbers  $0 = t_{i_0} < t_{i_1} < \dots < t_{i_k} < \dots$ , where  $t_{i_k} \rightarrow +\infty$  as  $k \rightarrow +\infty$ , such that  $\nu(t_{i_k}) \in \sigma_{i_k}$ ,  $\sigma_{i_j} \cap \sigma_{i_{j+1}} \neq \emptyset$ ,  $\text{int } \sigma_{i_j} \cap \text{int } \sigma_{i_{j+1}} = \emptyset$  and for any  $t > t_{i_k}$ ,  $\nu(t)$  does not belong to  $\bigcup_{j \leq k} \sigma_{i_j}$ . Denote by  $l_k$  the piece of straight line joining the points  $\nu(t_{i_k})$  and  $\nu(t_{i_{k+1}})$ ,  $k \in \mathbb{Z}^+$ . By construction, the sequence of the points  $\{\nu(t_{i_k})\}$  tends to  $a$  as  $k \rightarrow +\infty$ . Then the set  $\tilde{\gamma}_{a_0a} = \bigcup_{i \in \mathbb{Z}^+} l_i \cup \{a\}$  is a desired curve.

As  $\hat{F}^u$  is a continuous foliation consisting of smooth curves which are

transversal to disk  $Q$  there is a number  $N > 0$  such that for any  $c \in [-N, N]$  a plain  $P_c$  given by equation  $z = c$  intersects any leaf of the foliation  $\hat{F}^u$  in exactly one point.

For any point  $x \in \lambda$ , denote by  $\tilde{L}_x^u$  the closed arc such that:

1)  $\tilde{L}_x \subset L_x^u$ , where  $L_x^u$  is the leaf of the foliation  $\hat{F}^u$  passing through the point  $x$ ;

2)  $\tilde{L}_x$  lyes between the plains  $P_{-N} : z = -N$ ,  $P_N : z = N$ .

Then the arc  $\tilde{L}_{\nu(t_{i_k})}^u$  can be represented by equations:

$$x = x_k(z), y = y_k(z), z = z, z \in [-N, N].$$

Let us notice that by construction, the point  $\nu(t_{i_k})$  has the coordinates  $(x_k(0), y_k(0), 0)$ .

Denote by  $S_k$  the disk represented by the next equations:

$$x = x_k(z) + s_k(x_{k+1}(z) - x_k(z)), y = y_k(z) + s_k(y_{k+1}(z) - y_k(z)), z = z, z \in [-N, N], s_k \in [0, 1].$$

$$\text{Put } S_{a_0a} = \bigcup_{k \in \mathbb{Z}^+} S_k \cup \tilde{L}_a^u.$$

By construction,  $S_{a_0a}$  is a piecewise smooth disk. Using the piecewise linear curve  $\tilde{\gamma}_{bb_0}$  we can construct a piecewise smooth disk  $S_{bb_0}$  which is similar to  $S_{a_0a}$ .

$$\text{Put } S_{ab} = \bigcup_{x \in \lambda_1} \tilde{L}_x^u, S = S_{a_0a} \cup S_{bb_0} \cup S_{ab}, \nu_{-N} = S \cap P_{-N}, \nu_N = S \cap P_N.$$

It is not difficult to found the smoothly embedded closed disks  $S_1, S_2$  such that:

1)  $S_1, S_2$  transversely intersect any plain  $P_c$ ,  $c \in [-N, N]$ ;

2)  $\text{int } S_i \cap \text{int } S = \emptyset$ ,  $i = 1, 2$ ;

$$S_i \cap S = \tilde{L}_{a_0}^u \cup \tilde{L}_{b_0}^u$$

3)  $\text{int } S_1 \cap \text{int } S_2 = \emptyset$ ,

$$S_1 \cap S_2 = \tilde{L}_{a_0}^u \cup \tilde{L}_{b_0}^u;$$

4) the boundary of the disk  $S_i$  consists of the curves  $\tilde{L}_{a_0}^u, \tilde{L}_{b_0}^u$  and the curves  $\nu_{-N}^i = S_i \cap P_{-N}$ ,  $\nu_N^i = S_i \cap P_N$ ,  $i = 1, 2$ .

5) the curves  $\nu_{-N}^1, \nu_{-N}^2$  ( $\nu_N^1, \nu_N^2$ ) form the boundary of the closed disk  $D_{-N} \subset P_{-N}$  ( $D_N \subset P_N$ ) which contains the curve  $\nu_{-N}$  ( $\nu_N$ ) dividing the disk  $D_{-N}$  ( $D_N$ ) on two disks:  $D_{-N}^1$  bounded by the curves  $\nu_{-N}, \nu_{-N}^1$  and  $D_{-N}^2$  bounded by the curves  $\nu_{-N}, \nu_{-N}^2$  ( $D_N^1$  bounded by the curves  $\nu_N, \nu_N^1$  and  $D_N^2$  bounded by the curves  $\nu_N, \nu_N^2$ ).

Denote by  $B_i$  ( $i = 1, 2$ ) the closed set bounded by the union  $S \cup S_i \cup D_{-N}^i \cup D_N^i$ .

As intersection of the set  $B_i \cap P_c$  is homeomorphic to the standard disk  $D_0$ , then  $B_i$  is homeomorphic to  $D_0 \times [-N, N]$  and consequently  $B_i$  is homeomorphic to the standard ball  $B_0 : \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$ .

Then there is a neighborhood  $U_S$  of the disk  $S$  such that  $U_S \cap B_1$  ( $U_S \cap B_2$ ) is homeomorphic to  $D_0 \times [-1, 0]$  ( $D_0 \times [0, 1]$ ). Thus there are a neighborhood  $U_O$  of the point  $O$  and homeomorphism  $h_O : \overline{U_O} \rightarrow \mathbb{R}^3$  such that  $h_O(U_O \cap \hat{D})$  is the standard closed disk  $D_0$ .

The set  $B_z = g^{-1}(U_O)$  is a neighborhood of the point  $z$  in  $M^3$  and the map  $h_z = h_O \circ g : B_z \rightarrow \mathbb{R}^3$  satisfies to the following condition:  $h_z(B_z \cap V_0(z))$  is the standard closed disk  $D_0$ . It means that the surface  $M_i^2$  is tame. The lemma is proved.  $\square$

### 3 Proof of theorem 2

*Proof of theorem 2* follows from lemmas 4 and 5 which will be proved below.

For any point  $x \in M_i^2$  put  $L_i^s(x) = W^s(x) \cap M_i^2$  and denote  $F_i^s = \bigcup_{x \in M_i^2} L_i^s(x)$ . It follows from the local structure of direct product and the

proof of lemma 2 that there is  $\alpha > 0$  such that for any point  $z \in M_i^2$  there is the neighborhood  $V_0(z)$  such that:

1)  $W_\alpha^u(z) \subset V_0(z)$  and for any point  $y \in W_\alpha^u(z)$  the intersection  $W_\alpha^s(y) \cap V_0(z)$  consists of a simple curve  $\gamma_y^s$ ;

2) the family of curves  $\bigcup_{y \in W_\alpha^u(z)} \gamma_y^s$  is a continuous foliation on the neighborhood  $V_0(z)$ , that is there is a homeomorphism  $q_z : V_0(z) \rightarrow \mathbb{R}^2$  mapping the family  $\bigcup_{y \in W_\alpha^u(z)} \gamma_y^s$  to the set of straight lines parallel to the axes  $Ox$  on the plain  $\mathbb{R}^2$ .

3) any curve  $\gamma_y^s$  is locally section (in topological sense) to the foliation  $F_i^u$ .

As  $\mathcal{B}_i = M_i^2$  and for any point  $x \in \mathcal{B}_i$  the intersection  $W^s(x) \cap \mathcal{B}_i$  is dense in  $\mathcal{B}_i$ , then any leaf of the foliation  $F_i^s$  is dense in  $M_i^2$ . Thus the foliation  $F_i^s$  is a transitive foliation without singularities on the torus  $M_i^2$ .

Let us represent the torus  $M_i^2$  as the factor space  $\mathbb{R}^2/\Gamma$ , where  $\Gamma$  is a discrete group of motions  $\gamma_{m,n}$  of the plain  $\mathbb{R}^2$  given by the formulas  $\gamma_{m,n} : \bar{x} = x + m, \bar{y} = y + n, m, n \in \mathbb{Z}$ . Denote by  $\pi : \mathbb{R}^2 \rightarrow M_i^2$  the natural projection and  $g_*$  automorphism of the group  $\Gamma$  induced by diffeomorphism  $g$  ( $g = f^k$ ). Let us notice that automorphism  $g_*$  can be given by the following

way. Let  $\bar{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the covering homeomorphism for  $g : M_i^2 \rightarrow M_i^2$ , that is  $\pi \circ \bar{g} = g \circ \pi$ , and suppose that  $\bar{g}$  is given by the formulas:  $\bar{g} : \bar{x} = g_1(x, y), \bar{y} = g_2(x, y)$ . Then for any  $\gamma_{m,n} \in \Gamma$ , one can put  $g_*(\gamma_{m,n}) = \gamma_{m',n'}$ , where  $m' = g_1(m, n) - g_1(0, 0), n' = g_2(m, n) - g_2(0, 0)$ .

**Lemma 4** *The automorphism  $g_*$  is hyperbolic, that is the eigenvalues  $\lambda_1, \lambda_2$  of the matrix  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$  which induces automorphism  $g_*$  satisfies to the condition  $|\lambda_1| < 1, |\lambda_2| > 1$ .*

*Proof.* As  $F_i^u$  and  $F_i^s$  are transitive and transversal foliations on the torus  $M_i^2$  they form transitive 2-web on  $M_i^2$ . According to [3] (theorem 1), there are different irrational numbers  $\mu^u, \mu^s$  (which are Poincare rotation numbers of the foliation  $F_i^u$  and  $F_i^s$  respectively) and homeomorphism  $\varphi : M_i^2 \rightarrow M_i^2$  such that  $\varphi$  maps the foliations  $F_i^u$  and  $F_i^s$  to the linear foliation  $L_{\mu^u}$  and  $L_{\mu^s}$  respectively ( $L_{\mu^\sigma}$  is the image under the projection  $\pi$  of the foliation  $\bar{L}_{\mu^\sigma}$  any leaf of whose is given by the equation  $y = \mu^\sigma x + c, \sigma \in \{u, s\}, c \in \mathbb{R}^1$ ).

Denote by  $\bar{F}_i^\sigma$  the foliation on  $\mathbb{R}^2$  which is covering for the foliation  $F_i^\sigma$  ( $\sigma \in \{s, u\}$ ). Then there is a homeomorphism  $\bar{\varphi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  covering for the homeomorphism  $\varphi$  mapping the foliations  $\bar{F}_i^u$  and  $\bar{F}_i^s$  to the linear foliation  $\bar{L}_{\mu^u}$  and  $\bar{L}_{\mu^s}$  respectively. It follows from there that any leaf  $l^u$  of the foliation  $\bar{F}_i^u$  intersects any leaf  $l^s$  of the foliation  $\bar{F}_i^s$  at exactly one point.

Let us show now that the automorphism  $g_*$  is hyperbolic. Suppose the contrary. Let  $p \in Per(g)$  be a periodic points of a period  $m \geq 1$ . Without lost of generality, we can suppose that  $p = \pi(O)$  (where  $O$  is origin of the coordinate system on Euclidean plain). Denote by  $\bar{g}_m$  a covering homeomorphism for  $g^m$  such that  $\bar{g}_m(O) = O$ . The set  $\mathcal{O} = \bigcup_{\gamma \in \Gamma} \gamma(O)$  is a lattice

on the plain  $\mathbb{R}^2$ . The matrix  $\mathbf{A}^m$  defines the automorphism  $g_*^m$ . Denote by  $\mathcal{A}_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the linear map determined by the matrix  $\mathbf{A}^m$ . It follows from the definition of  $g_*$  that  $\bar{g}_m|_{\mathcal{O}} = \mathcal{A}_m|_{\mathcal{O}}$ . As a module of the eigenvalues of the matrix  $A^m$  is equal to 1 then there is a periodic point  $O_1 \in \mathcal{O}$  ( $O_1 \neq O$ ) of some period  $l \geq 1$  of the map  $\mathcal{A}_m|_{\mathcal{O}}$ . Consequently  $O_1$  is a periodic points of the diffeomorphism  $\bar{g}_m$ . Then the points  $O, O_1$  are fixed points of the diffeomorphism  $\bar{g}_m^l$ .

Denote by  $l_O^u$  ( $l_{O_1}^s$ ) the leaf of the foliation  $\bar{F}_i^u$  ( $\bar{F}_i^s$ ) passing through the point  $O$  ( $O_1$ ). As  $l_O^u \cap l_{O_1}^s \neq \emptyset$  and  $l_O^u, l_{O_1}^s$  are invariant unstable and stable manifold of the fixed saddle (in topological sense) points  $O, O_1$  respectively then there are infinitely many heteroclinic points of the diffeomorphisms  $\bar{g}_m^l$

belonging to the intersection  $l_O^u \cap l_{O_1}^s$ . We get contradicts with the fact that  $l_O^u \cap l_{O_1}^s$  consists of exactly one point. The lemma is proved.  $\square$

Denote by  $\mathcal{G}$  the linear automorphism of the torus  $M_i^2$  such that  $\mathcal{G}_* = g_*$ . According to [9] (proposition 2.1), there is a continuous homotopic to identity map  $h : M_i^2 \rightarrow M_i^2$  such that  $\mathcal{G}h = hg$ .

**Lemma 5** *The map  $h$  is a homeomorphism.*

*Proof.* Let  $\bar{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a covering map for  $h$ . Let us divide the proof of lemma into three steps.

*Step 1.* Let us show that if points  $\bar{x}, \bar{y} \in \mathbb{R}^2$  ( $\bar{x} \neq \bar{y}$ ) belong to the same leaf  $l^\sigma$  of the foliation  $\bar{F}_i^\sigma$ , then  $\bar{h}(\bar{x}) \neq \bar{h}(\bar{y})$ ,  $\sigma \in \{s, u\}$ .

Consider for definiteness the case  $\sigma = u$  (for  $\sigma = s$ , the proof is similar) and suppose the contrary, that is there are a leaf  $l^u$  of the foliation  $\bar{F}_i^u$  and points  $\bar{x}, \bar{y} \in l^u$  such that  $\bar{h}(\bar{x}) = \bar{h}(\bar{y})$ . Put  $x = \pi(\bar{x})$ ,  $y = \pi(\bar{y})$ ,  $L^u = \pi(l^u)$  and  $[x, y]^u \subset L^u$  is the closed arc with endpoints  $x, y$ . Let  $p$  be any periodic point of some period  $l \geq 1$  of the restriction of diffeomorphism  $g$  to  $M_i^2$ . Denote by  $L_p^s$  a leaf of the foliation  $F_i^s$  passing through the point  $p$ . We have two possibilities:

- a) the point  $p$  belongs to  $[x, y]^u$ ;
- b) the point  $p$  does not belong to  $[x, y]^u$ .

As the leaf  $L_p^s$  is dense on the surface  $M_i^2$ , then in the case b) there is a point  $v \in L_p^s \cap (x, y)^u$ .

Consequently there are two cases:

- $\bar{a}$ ) There is a point  $\bar{p} \in \pi^{-1}(p)$  belonging to the arc  $[\bar{x}, \bar{y}]^u \subset l^u$ ;
- $\bar{b}$ ) There are a point  $\bar{p} \in \pi^{-1}(p)$  and a point  $\bar{v} \in \pi^{-1}(v)$  such that  $\bar{p}$  and  $\bar{v}$  belong to the same leaf of the foliation  $\bar{F}_i^s$  and the point  $\bar{v}$  belong to the arc  $(\bar{x}, \bar{y})^u \subset l^u$ .

Let us consider a covering diffeomorphism  $\bar{g}_l$  for the diffeomorphism  $g^l$  such that  $\bar{g}_l(\bar{p}) = \bar{p}$ . Then in the both cases  $\bar{a}$ ) and  $\bar{b}$ ) we have  $\rho(\bar{g}_l^n(\bar{x}), \bar{g}_l^n(\bar{y})) \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

Let us notice that according to [9] (lemma 3.4), the map  $\bar{h}$  is proper<sup>4</sup>. Consequently, according to [8], there is a number  $r > 0$  such that for any points  $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^2$  satisfying a condition  $\bar{h}(\bar{x}_1) = \bar{h}(\bar{x}_2)$  an inequality  $\rho(\bar{x}_1, \bar{x}_2) < r$  is fulfilled<sup>5</sup>. As  $\mathcal{G}^l \circ h = h \circ g^l$  and  $h$  is homotopic to identity then there is a covering map  $\bar{\mathcal{G}}_l$  for the linear diffeomorphism  $\mathcal{G}^l$  such that  $\bar{\mathcal{G}}_l \circ \bar{h} = \bar{h} \circ g^l$ . Then

<sup>4</sup>a map  $\bar{h}$  is called proper if pre-image of a compact set is a bounded set.

<sup>5</sup>For convenience we repeat here arguments from [8]. Indeed, as the map  $\bar{h}$  is proper, then there is  $r > 0$  such that pre-image of a fundamental domain  $\Pi$  of the group  $\Gamma$  belongs

for any  $n \in \mathbb{Z}$  we have  $\bar{h}(\bar{g}^n(\bar{x})) = \bar{\mathcal{G}}_l^n(\bar{h}(\bar{x})) = \bar{\mathcal{G}}_l^n(\bar{h}(\bar{y})) = \bar{h}(\bar{g}^n(\bar{y}))$ . Consequently,  $\rho(\bar{g}^n(\bar{x}), \bar{g}^n(\bar{y})) < r$ . But it is impossible as  $\rho(\bar{g}_l^n(\bar{x}), \bar{g}_l^n(\bar{y})) \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

*Step 2.* Let us show that for any point  $\bar{x} \in \mathbb{R}^2$  the properties  $\bar{h}(l_{\bar{x}}^\sigma) = w^\sigma(\bar{h}(\bar{x}))$  is fulfilled, where  $l_{\bar{x}}^\sigma$  is the leaf of the foliation  $\bar{F}_i^\sigma$  and  $w^\sigma(\bar{x})$  is the straight line (passing through the point  $\bar{x}$ ) which is a pre-image of invariant manifold  $W^\sigma(\pi(\bar{x}))$  of the linear hyperbolic automorphism  $\mathcal{G}$ .

Consider for definiteness a case  $\sigma = s$  (in the case  $\sigma = u$  the proof is similar). First let us show that  $\bar{h}(l_{\bar{x}}^s) \subset w^s(\bar{h}(\bar{x}))$ . Let  $\bar{y}$  be any point belonging to  $l_{\bar{x}}^s$  ( $\bar{y} \neq \bar{x}$ ). Put  $x = \pi(\bar{x})$ ,  $y = \pi(\bar{y})$ . As  $\lim_{n \rightarrow +\infty} d(g^n(x), g^n(y)) \rightarrow 0$  ( $d$  is a metric on the torus  $M_i^2$ ) then by continuity of the map  $h$  we have  $\lim_{n \rightarrow +\infty} d(h(g^n(x)), h(g^n(y))) = \lim_{n \rightarrow +\infty} d(\mathcal{G}^n(h(x)), \mathcal{G}^n(h(y))) = 0$ . It follows from there that  $h(y) \subset W^s(h(x))$ . As  $\bar{h}$  is a covering map for  $h$  then  $\bar{h}(l_{\bar{x}}^s) \subset w^s(\bar{h}(\bar{x}))$ .

Let us show now that  $\bar{h}(l_{\bar{x}}^s) = w^s(\bar{h}(\bar{x}))$ . Suppose the contrary. As  $\bar{h}(w^s(\bar{x}))$  is a connected set which contains the point  $\bar{h}(\bar{x})$  and belongs to the straight line  $w^s(\bar{h}(\bar{x}))$ , then the image under the map  $\bar{h}$  of (at least) one component of the set  $l_{\bar{x}}^s \setminus \bar{x}$  is a bounded set on the straight line  $w^s(\bar{h}(\bar{x}))$ . It contradicts to the fact that the map  $\bar{h}$  is proper.

*Step 3.* Let us show that the map  $h$  is a homeomorphism. It follows from above arguments:

1) any point  $\bar{x}$  of the plain  $\mathbb{R}^2$  is a unique point of the intersections  $l_{\bar{x}}^s \cap l_{\bar{x}}^u$  and  $w^s(\bar{x}) \cap w^u(\bar{x})$ ;

2) the restriction of the map  $\bar{h}$  to each curve  $l_{\bar{x}}^s$ ,  $l_{\bar{x}}^u$  is a one-to-one map on  $w^s(\bar{h}(\bar{x}))$ ,  $w^u(\bar{h}(\bar{x}))$  respectively.

It follows from 1) and 2) that the map  $\bar{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is one-to-one. Then the map  $h : M_i^2 \rightarrow M_i^2$  is a continuous one-to-one map and consequently is a homeomorphism.

The lemma is completely proved.  $\square$

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to the open disk  $D_{\frac{r}{2}} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < \frac{r^2}{4}\}$ . Let  $\gamma \in \Gamma$  such that  $\gamma(\bar{h}(\bar{x}_1)) \in \Pi$ . Then as the map  $h$  is homotopic to identity and  $\bar{h}(\bar{x}_1) = \bar{h}(\bar{x}_2)$  we get  $\gamma(\bar{h}(\bar{x}_1)) = \gamma(\bar{h}(\bar{x}_2))$ ,  $\bar{h}(\gamma(\bar{x}_1)) = \bar{h}(\gamma(\bar{x}_2))$ . It follows from there that  $\rho(\gamma(\bar{x}_1), \gamma(\bar{x}_2)) < r$ . As the map  $\gamma$  is an isometry of  $\mathbb{R}^2$  then  $\rho(\bar{x}_1, \bar{x}_2) < r$ .

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